INDIAN STATISTICAL INSTITUTE
Back-Paper Exam
Algebra-II
2017-2018

Total marks: 100
Time: 3 hours

Answer all questions.

(1) Let V, W be finite dimensional vector spaces over $\mathbb{R}$ and let $T : V \longrightarrow W$ be a linear transformation. Show that

$$dim(V) = dim(Ker(T)) + dim(Range(T)).$$

[15]

(2) Let $\mathcal{P}_2$ denote the space of all real polynomials of degree atmost 2. Consider the linear transformation $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ given by $T(f)(x) = f(2x-1)$.
(i) Find the matrix of T with respect to the basis $\{1, x, x^2\}$.
(ii) Find the eigenvalues of T.
(iii) Is T diagonalizable? Give reasons.
(iv) Compute the eigenspaces of T. [6+6+3+10]

(3) Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. [20]

(4) Let X be the plane in $\mathbb{R}^3$ spanned by vectors

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

(i) Find an orthonormal basis for X.
(ii) Extend it to an orthonormal basis for $\mathbb{R}^3$. [10+10]

(5) (i) State spectral theorem for $n \times n$ real symmetric matrices.
(ii) Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix. [4+16]